Theory of ODEs

Reference :

" Qualitative Theory of Ordinary differential Equations "

by

Brauer and Nohel

Symbus :

1- Linear systems « General solution »

2- Linear systems « Jordan forms, phase plane »

3- stability at linear systems

4- Linear systems with periodic coefficients « Floequet theorem »

5- stability by linearization

6- Liapanov theory of stability « Liapunov's theorem, quadratic forms »

1- Linear Systems

$$x \in \mathbb{R}^n$$

$$x(t) : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^n$$

$$x(t) = (x_1(t), x_2(t), \cdots, x_n(t))$$

$$\dot{x} = \frac{dx}{dt} \quad , \quad \dot{x} = (\dot{x}_1, \dot{x}_2, \dot{x}_3, \cdots, \dot{x}_n)$$

The linear system of differential equations can be written as:

$$\dot{x} = A(t)x + B(t) \qquad \cdots \quad (1)$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Therefore equation (1) can be expressed as:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Hence

$$\left.\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_1 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_2 \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_n \end{aligned}\right\} \qquad \cdots \quad (2)$$

Now, we want to find the function that satisfy equation (1).

Theorem:-

If A and B are continuous on $I \subseteq \mathbb{R}$, then for any $t_0 \in I$ and $x_0 \in \mathbb{R}^n$, $\exists!$ solution ϕ for $\dot{x} = Ax + B$ s.t: $\phi(t_0) = x_0$, the solution is defined on I.

Theorem :-

The set of all solutions of $\dot{x} = Ax$, forms a vector space of dimension n.

Proof:- we want to prove two things:

1- The set of solutions of $\dot{x} = Ax$, forms a vector space.

2- The dimension of this vector space is n.

let u_1 and u_2 be two solutions of $\dot{x} = Ax$, thus we have:

$$\dot{u}_1 = Au_1, \quad \dot{u}_2 = Au_2$$

To prove that

① $(u_1 + u_2)' = A(u_1 + u_2)$

$$(u_1 + u_2)' = \dot{u}_1 + \dot{u}_2 = Au_1 + Au_2 = A(u_1 + u_2)$$

② if $\dot{u} = Au$, we want to prove $(ku)' = A(ku)$

$$(ku)' = k\dot{u} = kAu = A(ku)$$

Therefore, the set of solutions of $\dot{x} = Ax$, form a vector space.

Now, we need to prove that the dimension of the vector space is n.

Consider $t_0 \in I$.

let ϕ_1 be the solution s.t. : $\phi_1(t_0) = e_1$

ϕ_2 : : : : : $\phi_2(t_0) = e_2$

$\vdots$

ϕ_n : : : : : $\phi_n(t_0) = e_n$

where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix},$$

if $c_1 \phi_1(t) + c_2 \phi_2(t) + \cdots + c_n \phi_n(t) \equiv 0$, $\forall\, t \in I$

put $t = t_0$

Then

$$c_1 \phi_1(t_0) + c_2 \phi_2(t_0) + \cdots + c_n \phi_n(t_0) = 0$$

So

$$c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0.$$

Because $e_1, e_2, \ldots, e_n$ are linearly independent.

Therefore $\phi_1, \phi_2, \ldots, \phi_n$ are linearly independent.

If $\underline{\Phi}(t)$ is a solution (to prove span)

$\underline{\Phi}(t_0) \in \mathbb{R}^n$, since $\langle e_1, e_2, \ldots, e_n \rangle = \mathbb{R}^n$

$\exists\, c_1, c_2, \ldots, c_n$ s.t. : $\underline{\Phi}(t_0) = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$

Define $\quad \psi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t)$

$\psi(t)$ is a Solution, $\psi(t_0) = c_1\phi_1(t_0) + c_2\phi_2(t_0) + \cdots + c_n\phi_n(t_0)$

then $\quad \psi(t_0) = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$

hence

$$\Phi(t) = \psi(t), \quad \forall\ t \in I$$

$$\Phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \cdots + c_n\phi_n(t), \quad \forall\ t \in I$$

$\therefore \quad \langle \phi_1, \phi_2, \cdots, \phi_n \rangle =$ The vector space of all solutions of $\dot{x} = Ax$

Therefore, the dimension of the vector space is n.

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Remarks :- ① If $\phi_1, \phi_2, \cdots, \phi_n$ are solutions of $\dot{x} = Ax$, then we say that $\{\phi_1, \phi_2, \cdots, \phi_n\}$ is a solution set.

② If $\phi_1, \phi_2, \cdots, \phi_n$ are linearly independent solutions of $\dot{x} = Ax$, then we say that $\{\phi_1, \phi_2, \cdots, \phi_n\}$ is a fundamental set of solutions.

usually written as

$$(\phi_1\ \phi_2\ \cdots\ \phi_n)_{n\times n} = \text{Solution matrix}$$

$$= \text{Fundamental solu. matrix denoted by } \Phi(t)$$

Hence

$$\Phi(t) = (\phi_1\ \phi_2\ \cdots\ \phi_n)_{n\times n}$$

Theorem (1) :-

A solution matrix $\underline{\Phi}(t)$ is a fundamental solution matrix if and only if $\det \underline{\Phi}(t) \neq 0, \ \forall t \in I$.

Theorem (2) :-

$$\det \underline{\Phi}(t) = \det \underline{\Phi}(t_0) \cdot \exp \int_{t_0}^{t} \operatorname{tr}(A(s))\, ds, \ \forall t \in I$$

where $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}$

Proof of theorem (1) :-

If $\underline{\Phi}(t)$ is a F.M.S., T.P. $\det \underline{\Phi}(t) \neq 0, \ \forall t \in I$.

If $\det \underline{\Phi}(t) = 0$

$\Rightarrow \exists \ t_0 \in I$ s.t. :

$\phi_1(t_0), \phi_2(t_0), \ldots, \phi_n(t_0)$ are linearly dependent vectors in $\mathbb{R}^n$.

$\exists \ c_1, c_2, \ldots, c_n$ (« $c_i \neq 0$, for some i »), s.t :

$$c_1 \phi_1(t_0) + c_2 \phi_2(t_0) + \cdots + c_n \phi_n(t_0) = 0$$

We define $\phi(t) = c_1 \phi_1(t) + c_2 \phi_2(t) + \cdots + c_n \phi_n(t)$

$\phi(t)$ is a solution and $\phi(t_0) = 0 \longrightarrow \phi(t) = 0, \ \forall t$.

Therefore

$$c_1 \phi_1(t) + c_2 \phi_2(t) + \cdots + c_n \phi_n(t) = 0, \ \forall t, \ c_i \neq 0 \text{ for some } i$$

But $\phi_1(t), \phi_2(t), \ldots, \phi_n(t)$ are linearly independent (Because $\underline{\Phi}(t) = (\phi_1(t) \ \phi_2(t) \cdots \phi_n(t))$ is a F.M.S.) C!

Hence

$$\det \Phi(t) \neq 0$$

$\Leftarrow$ Conversly

Now if $\det \Phi(t) \neq 0$, T.P. $\Phi(t)$ is a F.M.S.

since $\det \Phi(t) \neq 0, \forall t \in I$

then

$$c_1 \phi_1(t) + c_2 \phi_2(t) + \cdots + c_n \phi_n(t) = 0, \quad \forall t \in I$$

implies

$$\begin{pmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0$$

i.e

$$\Phi(t) C = 0$$

fix $t = t_0$

$$\Phi(t_0) C = 0 \quad \text{and since } \det \Phi(t_0) \neq 0$$

Then

$$c_1 = c_2 = \cdots = c_n = 0$$

Therefore $\begin{pmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \end{pmatrix}$ are linearly independent

since $\phi_1(t), \phi_2(t), \cdots, \phi_n(t)$ are solutions and we have been proved above that they are linearly independent

Then

$$\Phi(t) \text{ is a F.M.S.}$$

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Proof of theorem (2):-

First, we want to prove that

$$(\det \Phi(t))' = (\det \Phi(t)) \operatorname{tr}(A(t))$$

if $n=2$, we have

$$\Phi(t) = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \phi_{11} \\ \phi_{21} \end{pmatrix}, \quad \begin{pmatrix} \phi_{12} \\ \phi_{22} \end{pmatrix}' = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \phi_{12} \\ \phi_{22} \end{pmatrix}$$

$$\phi_{11}' = a_{11}\phi_{11} + a_{12}\phi_{21}$$

$$\phi_{21}' = a_{21}\phi_{11} + a_{22}\phi_{22}$$

$$\phi_{12}' = a_{11}\phi_{12} + a_{12}\phi_{22}$$

$$\phi_{22}' = a_{21}\phi_{12} + a_{22}\phi_{22}$$

$$(\det \Phi(t))' = (\phi_{11}\phi_{22} - \phi_{12}\phi_{21})'$$

$$= \phi_{11}\phi_{22}' + \phi_{11}'\phi_{22} - \phi_{12}\phi_{21}' - \phi_{12}'\phi_{21}$$

$$= \phi_{11}(a_{21}\phi_{12} + a_{22}\phi_{22}) + (a_{11}\phi_{11} + a_{12}\phi_{21})\phi_{22} -$$

$$- \phi_{12}(a_{21}\phi_{11} + a_{22}\phi_{21}) - (a_{11}\phi_{12} + a_{12}\phi_{22})\phi_{21}$$

$$= a_{21}\phi_{11}\phi_{12} + a_{22}\phi_{11}\phi_{22} + a_{11}\phi_{11}\phi_{22} + a_{12}\phi_{21}\phi_{22} - a_{21}\phi_{12}\phi_{11}$$

$$- a_{22}\phi_{12}\phi_{21} - a_{11}\phi_{12}\phi_{21} - a_{12}\phi_{22}\phi_{21}$$

$$= a_{11}(\phi_{11}\phi_{22} - \phi_{12}\phi_{21}) + a_{22}(\phi_{11}\phi_{22} - \phi_{12}\phi_{21})$$

$$= (\phi_{11}\phi_{22} - \phi_{12}\phi_{21})(a_{11} + a_{22})$$

Thus

$$(\det \Phi(t))' = \det \Phi(t) \operatorname{tr}(A(t))$$

$$\frac{(\det \Phi(t))'}{\det \Phi(t)} = \operatorname{tr} A(t)$$

$$\int_{t_0}^{t} \frac{(\det \Phi(s))'}{\det \Phi(s)} = \int_{t_0}^{t} \operatorname{tr} A(s)\, ds$$

$$\Rightarrow \ln \det \Phi(s) \Big|_{t_0}^{t} = \int_{t_0}^{t} \operatorname{tr} A(s)\, ds$$

$$\ln \det \Phi(t) - \ln \det \Phi(t_0) = \int_{t_0}^{t} \operatorname{tr} A(s)\, ds$$

$$\ln \det \Phi(t) = \ln \det \Phi(t_0) + \int_{t_0}^{t} \operatorname{tr} A(s)\, ds$$

$$\exp(\ln \det \Phi(t)) = \exp\left[\ln \det \Phi(t_0) + \int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right]$$

$$\det \Phi(t) = \exp(\ln \det \Phi(t_0)) \cdot \exp\left[\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right]$$

$$\det \Phi(t) = \det \Phi(t_0) \cdot \exp\left[\int_{t_0}^{t} \operatorname{tr} A(s)\, ds\right]$$

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Remarks :-

Consider $\dot{x} = Ax$

① If $\Phi(t)$ is a fundamental matrix, then any solution $\phi(t)$ is of form $\Phi(t)C$.

② If $\Phi(t)$ is a fundamental matrix and $C_{n\times n}$ is a nonsingular matrix, then $\Phi(t)C$ is also a fundamental matrix.

Proof :-

① $\Phi(t) = (\phi_1(t) \quad \phi_2(t) \quad \phi_n(t))$

$\exists\ c_1, c_2, \dots, c_n$ s.t : $\phi(t) = c_1\phi_1(t) + c_2\phi_2(t) + \dots + c_n\phi_n(t)$

$$= (\phi_1\ \phi_2 \cdots \phi_n)\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

$$= \Phi(t)C$$

Therefore $\phi(t) = \Phi(t)C$.

② $(\phi_1(t) \quad \phi_2(t) \cdots \phi_n(t))C =$

$$\begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & & & \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} =$$

$$= \Big(c_{11}\begin{pmatrix}\phi_{11}\\ \phi_{21}\\ \vdots \\ \phi_{n1}\end{pmatrix} + c_{21}\begin{pmatrix}\phi_{12}\\ \phi_{22}\\ \vdots \\ \phi_{n2}\end{pmatrix} + \cdots + c_{n1}\begin{pmatrix}\phi_{1n}\\ \phi_{2n}\\ \vdots \\ \phi_{nn}\end{pmatrix} \quad c_{12}\phi_{11} + c_{22}\phi_{12} + \cdots + c_{n2}\phi_{1n} \cdots \Big)$$

$\phi_1 \qquad \phi_2 \qquad \phi_n$

$$= \Big(c_{11}\phi_1 + c_{21}\phi_2 + \cdots + c_{n1}\phi_n \quad ---- \Big)$$

since every Coloumn is a linear combination of solutions of $\dot{x} = Ax$

then it is a the matrix $\Phi(t)C$ is a solution for $\dot{x} = Ax$.

$$\det(\Phi C) = \det\Phi \cdot \det C \neq 0$$

$\det\Phi$: F.M.S ← by thm (1) : $\det\Phi(t) \neq 0$

$\det C$: nonsingular matrix, $\det C \neq 0$

Therefore $(\Phi(t)C)$ is a Fundamental matrix solution

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Remark :- The solution of the nonhomogenous system

$$\dot{x} = A(t)x + B(t)$$

does not form a vector space.

Proof:- let u_1 and u_2 be two solutions of $\dot{x} = Ax + B$, then

$$\dot{u}_1 = Au_1 + B \;,\; \dot{u}_2 = Au_2 + B$$

To prove that $(u_1+u_2)' \overset{??}{=} A(u_1+u_2) + B$

$$(u_1+u_2)' = \dot{u}_1 + \dot{u}_2 = Au_1 + B + Au_2 + B$$
$$= A(u_1+u_2) + \underline{\underline{2B}}$$

Therefore it is not form a vector space.

Theorem :-

If $\Phi(t)$ is a fundamental matrix for the system $\dot{x} = Ax$, then the solution $\phi(t)$ of the nonhomogenous system $\dot{x} = A(t)x + B(t)$ satisfying $\phi(t_0) = c_0$ is given by:

$$\boxed{\phi(t) = \Phi(t)\Phi^{-1}(t_0)c_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)B(s)\,ds}$$

Proof:-

$$\phi'(t) = \phi'(t)\Phi^{-1}(t_0)c_0 + \Phi'(t)\int_{t_0}^{t}\Phi^{-1}(s)B(s)\,ds + \Phi(t)\Phi^{-1}(t)B(t)$$

$$= A\Phi(t)\Phi^{-1}(t_0)c_0 + A\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)B(s)\,ds + B(t)$$

$$= A\left[\Phi(t)\Phi^{-1}(t_0)c_0 + \Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)B(s)\,ds\right] + B(t)$$

$$= A\phi(t) + B(t)$$

$=//=$

Eigenvalues and Eigenvectors :-

$$Ax = \lambda x$$

x : eigenvector, λ : eigenvalues

Def:- The eigenvalues are the roots of the polynomial

$$|A - \lambda I| = 0$$

Remark :- If x is an eigenvector with λ an eigenvalue, then Kx is also an eigenvector with λ an eigenvalue.

Remark :- let $p(\lambda) = |A - \lambda I|$, then p has n-roots containing multiplicity

$$p(\lambda) = \mp (\lambda - \lambda_1)^{k_1} (\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_r)^{k_r}$$

$$k_1 + k_2 + \cdots + k_r = n$$

λ_1 is an eigenvalue of multiplicity k_1

λ_2 " " " " k_2

$\vdots$

λ_r " " " " k_r

Example :-

$$p(\lambda) = (\lambda - 1)^2 (\lambda + 2)^3$$

$\lambda_1 = -2$ with multiplicity 3

$\lambda_2 = 1$ " " 2.

H.W

Theorem :- If $v_1, v_2, \dots v_r$ are eigenvectors with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$, then $v_1, v_2, \dots, v_r$ are linearly independent.

Now, we want to Construct the Fundamental matrix of solutions of $\dot{x} = Ax$

If A has n distinct eigenvalues, $\lambda_1, \lambda_2, \dots \lambda_n$, then

$$\phi_1(t) = e^{\lambda_1 t} c_1 \,,\ \phi_2(t) = e^{\lambda_2 t} c_2 \,, \dots,\ \phi_n(t) = e^{\lambda_n t} c_n$$

where

$$Ac_i = \lambda c_i \,,\ c_i \neq 0 \,,\ i = 1, 2, \dots, n \,,$$

are linearly independent solution for $\dot{x} = Ax$.

Proof:-

$$\Phi(t) = (\phi_1(t), \dots \phi_n(t))$$

$$\phi_i(t) = e^{\lambda_i t} c_i$$

① $\dot{\phi}_i(t) = \lambda_i e^{\lambda_i t} c_i = e^{\lambda_i t} \lambda_i c_i = e^{\lambda_i t} A c_i = A e^{\lambda_i t} c_i = A \phi_i(t)$

② $\det \Phi(0) = \det (c_1\ c_2 \dots c_n) \neq 0$

{ since $c_1, c_2, \dots, c_n$ are linearly independent, then its determinant does not equal to zero, furthermore if the determinant does not equal to zero at a point then its not equal to zero at all the point of the interval }.

Remark :-

If $u(t) = u_1(t) + i u_2(t)$ is a solution for $\dot{x} = Ax$, then u_1, u_2 are solutions for $\dot{x} = Ax$.

$$\dot{u} = (u_1 + iu_2)' = A(u_1 + iu_2) = Au_1 + iAu_2$$

$$\dot{u}_1 = Au_1 \ , \ \dot{u}_2 = Au_2$$

Exponential of a Matrix :-

let $M_{n\times n}$ be an $n \times n$ matrix, then

$$e^{M} = \sum_{i=0}^{\infty} \frac{M^i}{i!} = I + M + \frac{M^2}{2!} + \cdots$$

H.W prove that the above series is convergent.

Now, Consider the linear systems with constant coefficients:

$$y' = Ay$$

Theorem :- The matrix

$$\Phi(t) = \exp(At)$$

is the fundamental matrix of $y' = Ay$ with $\Phi(0) = I$ on $-\infty < t < \infty$.

Proof:- That $\Phi(0)=I$ is obvious from

$$\exp M = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots + \frac{M^k}{k!} + \cdots$$

~~over $-\infty<t<\infty$~~.

Using the above equation with $M=At$ (well defined for $-\infty<t<\infty$ and every $n\times n$ matrix A), we have by differentiation the power series

$$(\exp At)' = A + \frac{A^2 t}{1!} + \frac{A^3 t^2}{2!} + \cdots + \frac{A^k t^{k-1}}{(k-1)!} + \cdots = A\exp At.$$

$-\infty<t<\infty$.

Therefore $\exp(At)$ is a solution matrix of $y'=Ay$.

Since $\det \Phi(0) = \det I = 1$, (Abel's formula) with $t_0=0$, gives

$$\det(\exp At) = \exp\left(\sum_{k=0}^{n} a_{kk}\right)t \neq 0, \quad -\infty<t<\infty$$

Therefore by theorem $\Phi(t)$ is a solution matrix of $y'=Ay$.

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Abel's formula:

$$\det \Phi(t) = \det \Phi(t_0) \exp\left[\int_{t_0}^{t} \sum_{j=1}^{n} a_{jj}(s)\,ds\right], \quad \forall t\in I.$$

Remarks

① $e^{M+N} = e^{M} \cdot e^{N}$ "if $MN = NM$", (N and M are Matrices)

② $e^{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}} = \begin{pmatrix} e^{\lambda_1} & 0\\ 0 & e^{\lambda_2}\end{pmatrix}$

③ $e^{\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}} = e^{\lambda}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

④ $e^{\begin{pmatrix}0 & -\beta\\ \beta & 0\end{pmatrix}} = \begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}$

⑤ $e^{TAT^{-1}} = T e^{A} T^{-1}$

Proof ②

$$e^{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}} = \sum_{i=0}^{\infty} \frac{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^i}{i!} = \sum_{i=0}^{\infty} \frac{\begin{pmatrix}\lambda_1^i & 0\\ 0 & \lambda_2^i\end{pmatrix}}{i!}$$

$$= \begin{pmatrix} \sum_{i=0}^{\infty} \frac{\lambda_1^i}{i!} & 0 \\ 0 & \sum_{i=0}^{\infty} \frac{\lambda_2^i}{i!}\end{pmatrix} = \begin{pmatrix} e^{\lambda_1} & 0\\ 0 & e^{\lambda_2}\end{pmatrix}$$

③ $e^{\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}} = e^{\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}$ $\left\{ \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix} \right\}$ (إبدال)

$$= e^{\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}} e^{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}} = \begin{pmatrix} e^{\lambda} & 0\\ 0 & e^{\lambda}\end{pmatrix}\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} + \frac{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}^2}{2!} + \cdots\right]$$

$$= e^{\lambda} I \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = e^{\lambda}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad \#$$

④
$$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}^2 = \begin{pmatrix} -\beta^2 & 0 \\ 0 & -\beta^2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}^3 = \begin{pmatrix} 0 & \beta^3 \\ -\beta^3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}^4 = \begin{pmatrix} \beta^4 & 0 \\ 0 & \beta^4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}^5 = \begin{pmatrix} 0 & -\beta^5 \\ \beta^5 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}^6 = \begin{pmatrix} -\beta^6 & 0 \\ 0 & -\beta^6 \end{pmatrix}$$

$$\vdots \qquad\qquad \vdots$$

Therefore
$$e^{\begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}} = \begin{pmatrix} 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} + \cdots & -\beta + \frac{\beta^3}{3!} - \frac{\beta^5}{5!} + \cdots \\ \beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \cdots & 1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \frac{\beta^6}{6!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

⑤ $$e^{TAT^{-1}} = \sum_{i=0}^{\infty} \frac{(TAT^{-1})^i}{i!} = \sum_{i=0}^{\infty} \frac{TA^iT^{-1}}{i!} = T \sum_{i=0}^{\infty} \frac{A^i}{i!} T^{-1} = Te^AT^{-1}$$

illustration

$$(TAT^{-1})^2 = TATATAT^{-1}$$
$$= TA^2T^{-1}$$

$$(TAT^{-1})^3 = TA^2TATAT^{-1}$$
$$= TA^3T^{-1}$$

and so on.

Example :- Find a fundamental matrix of $\dot{y} = Ay$ if

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Solution :- since

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \left[\text{check } \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right]$$

and since these two matrices commute, we have:

$$\underline{\Phi}(t) = e^{\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} t} = e^{\left[\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] t}$$

$$= e^{\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} t} \; e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t}$$

$$= \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix} \left[\underline{I} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 \frac{t^2}{2!} + \cdots \right]$$

But $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and the infinite series terminates after two terms.

Therefore,

$$e^{At} = \underline{\Phi}(t) = e^{\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} t} = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \right]$$

$$= e^{3t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\underline{\Phi}(t) = e^{3t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$=\!\!=\!\!\#$

Example :- Find a fundamental matrix of the system $y' = Ay$, where

$$A = \begin{pmatrix} 3 & 5 \\ -5 & 3 \end{pmatrix}.$$

Solu:

The eigenvalues of A are roots of the equation

$$|A - \lambda I| = 0 \quad \Rightarrow \quad \begin{vmatrix} 3-\lambda & 5 \\ -5 & 3-\lambda \end{vmatrix} = 0$$

So $\lambda_{1,2} = 3 \pm 5i$.

The eigenvector $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ corresponding to the eigenvalue $\lambda_1 = 3+5i$

must satisfy the linear homogeneous algebraic system

$$(A - \lambda_1 I)u = \begin{pmatrix} -5i & 5 \\ -5 & -5i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

Thus u_1, u_2, satisfy the system of equations

$$-iu_1 + u_2 = 0$$
$$-u_1 - iu_2 = 0$$

and therefore

$$u = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

is an eigenvector.

Similarly, the eigenvector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ corresponding to the eigenvalue

$\lambda_2 = 3-5i$ is found to be

$$v = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

Since $\lambda_1 = 3+5i$ and $\lambda_2 = 3-5i$ are eigenvalues of A and having $u_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $u_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$ are corresponding respective eigenvectors, it follows that

$e^{\lambda_1 t} c_1$ $\qquad$ $e^{\lambda_2 t} c_2$

$$\phi_1(t) = \exp(3+5i)t \begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } \phi_2(t) = \exp(3-5i)t \begin{pmatrix} i \\ 1 \end{pmatrix}$$

are solutions. Moreover, ϕ_1 and ϕ_2 are linearly independent and therefore

$$\Phi(t) = \begin{pmatrix} \exp(3+5i)t & i\exp(3-5i)t \\ i\exp(3+5i)t & \exp(3-5i)t \end{pmatrix}$$

is a fundamental matrix.

$\neq$

To determine the form e^{At} when A is an arbitrary matrix, we require the following result from linear algebra.

We note that the following method, which leads to the formula

$$\Phi(t) = \sum_{j=1}^{K} \exp(\lambda_j t) \left[\sum_{i=0}^{n_j - 1} \frac{t^i}{i!} (A - \lambda_j I)^i \right] v_j \quad \text{vectors} \quad \cdots \circledast$$

is always applicable, and includes the technique described above as a special case.

Let A be a (complex) n-by-n matrix. Compute $\lambda_1, \lambda_2, \cdots, \lambda_K$ the distinct eigenvalues of A with respective multiplicities $n_1, n_2, \cdots, n_K$ where $n_1 + n_2 + \cdots + n_K = n$.

Corresponding to each eigenvalue λ_j of multiplicity n_j consider the system of linear equations

$$(A - \lambda_j I)^{n_j} x = 0 \quad . (j = 1, 2, \cdots, K) \quad \cdots ①$$

The solutions of each such linear system obviously span a subspace of E_n which we call X_j $(j = 1, 2, \cdots, K)$.

The result from linear algebra needed tell us that for every $x \in E_n$ there exist unique vectors $x_1, x_2, \cdots, x_K$, where $x_j \in X_j$, such that

$$x = x_1 + x_2 + \cdots + x_K \qquad \cdots (2)$$

and $x_j \in X_j$ $(j = 1, 2, \cdots, K)$.

Remarks:-

(1) It is important to know that the linear algebraic system (1) has n_j linearly independent solutions so that the dimension of X_j is n_j.

(2) We note that if all the eigenvalues of A are distinct, that is if each $n_j = 1$ $(j = 1, 2, \cdots, K)$ and $K = n$, then the vectors $x_1, x_2, \cdots, x_n$ are suitable multiples of fixed eigenvectors that are linearly independent and span E_n.

In the language of linear algebra, E_n is the direct sum of subspaces $X_1, X_2, \cdots, X_K$,

$$E_n = X_1 \oplus X_2 \oplus \cdots \oplus X_K \qquad \cdots (3)$$

such that the dimension of X_j is n_j.

Remark :-

It may, in fact happen that in (1) $(A - \lambda_j I)^q x = 0$ for all x in X_j and $q < n_j$, as the following example demonstrates.

Example:- Consider the matrix

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

then $E_n = E_5$, and $\lambda = 4$ is the only eigenvalue, with multiplicity 5; that is $k=1$ (because there exist only one eigenvalue).

Since there is only one eigenvalue, no decomposition into subspaces is necessary, and $E_5 = X_1$.

According to the theory $n_1 = 5$, so that certainly

$$(A - 4I)^5 = 0 .$$

However, we can verify that

$$(A-4I)^3 = 0$$

but $(A-4I)^2 \neq 0$.

Therefore $q = 3 < n_1 = 5$.

To apply this theory to the linear system $\dot{y} = Ay$, we look for that solution $\underline{\Phi}(t)$ satisfying the initial condition

$$\underline{\Phi}(0) = \underline{\eta}$$

We know that

$$\underline{\Phi}(t) = e^{At}\underline{\eta}$$

Our object is to evaluate $e^{At}\eta$ explicitly, that is to see exactly what the components of Φ are.

We compute $\lambda_1, \lambda_2, \cdots, \lambda_K$, the eigenvalues of A of multiplicities $n_1, n_2, \cdots, n_K$ respectively.

We apply the theory of linear algebra to the initial condition or vector η and is accordance with (2), we have:

$$\eta = v_1 + v_2 + \cdots + v_K$$

where v_j is some suitable vector in the subspace X_j $(j = 1, 2, \cdots, K)$.

Since the subspace X_j is generated by the system (1), v_j must be the solution of (1).

Now $$e^{At}\eta = \sum_{j=1}^{K} e^{At} v_j$$

$$e^{At} v_j = \exp(\lambda_j t) \exp[(A - \lambda_j I)t] v_j$$

$$= \exp(\lambda_j t)\left[I + t(A - \lambda_j I) + \frac{t^2}{2!}(A - \lambda_j I)^2 + \cdots + \frac{t^{n_j - 1}}{(n_j - 1)!}(A - \lambda_j I)^{n_j - 1} \right] v_j$$

for sectors, where the series in parentheses terminates because v_j is solution of (1); thus the term $(A - \lambda_j I)^{n_j} v_j = 0$ and all subsequent terms are zero.

Applying the above calculations to the solution $\underline{\Phi}(t) = e^{At}\eta$ of $y' = Ay$, we have:

$$\underline{\Phi}(t) = e^{At}\eta = e^{At}\sum_{j=1}^{K} v_j = \sum_{j=1}^{K} e^{At} v_j$$

$$= \sum_{j=1}^{K} \exp(\lambda_j t)\left[I + (A-\lambda_j I)t + \cdots + \frac{t^{n_j-1}}{(n_j-1)!}(A-\lambda_j I)^{n_j-1}\right] v_j$$

Finally, the solution $\underline{\Phi}$ satisfying $\underline{\Phi}(0) = \eta$ is

$$\underline{\Phi}(t) = \sum_{j=1}^{K} \exp(\lambda_j t)\left[\sum_{i=0}^{n_j-1} \frac{t^i}{i!}(A-\lambda_j I)^i\right] v_j, \quad -\infty < t < \infty$$

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Example :- Solve the linear system $y' = Ay$, if

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$$

Also obtain a fundamental matrix.

Solu: the matrix A has an eigenvalue $\lambda = 3$ with multiplicity 2

In the above notation, $n_1 = 2$. Therefore only one subspace X_1 in this case ($X_1 = E_2$).

We readily calculate

$$A - 3I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

and we also that

$$(A-3I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so that eqn. (2) is satisfied for every vector in E_2.

Substituting into eqn. (*) with $\eta_1 = 2$, $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$, we find

$$\underline{x}(t) = e^{3t}\left[I + (A-3I)\right]\eta$$

and therefore

$$\underline{x}(t) = e^{3t}\left[I + \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} t\right]\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$$= e^{3t}\begin{bmatrix} \eta_1 + t(-\eta_1 + \eta_2) \\ \eta_2 + t(-\eta_1 + \eta_2) \end{bmatrix} \qquad \text{---(4)}$$

is the solution with $\underline{x}(0) = \eta$.

To construct a fundamental matrix, we say that e^{At} is a fundamental matrix.

So far, we have computed, in eqn. (4) $\exp(At)\eta$ for any an arbitrary constant vector η. But

$$\exp(At) = \exp At \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\exp(At)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \exp At \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

where the solution vectors $\exp(At)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\exp(At)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are found from equation (4).

By substituting first $\eta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and then $\eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively.

Therefore a fundamental matrix is

$$\Phi(t) = \exp At = e^{2t} \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$$

$$\#$$

Example:- Consider the system

$$x_1' = 3x_1 - x_2 + x_3$$

$$x_2' = 2x_1 + x_3$$

$$x_3' = x_1 - x_2 + 2x_3$$

which has the coefficient matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 2 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

Find the solution Φ satisfying the initial condition

$$\Phi(0) = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \eta$$

and also a fundamental matrix.

The characteristic polynomial of A is

$$|A-\lambda I| = (\lambda-1)(\lambda-2)^2$$

and therefore the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ with multiplicities $n_1 = 1$, $n_2 = 2$, respectively.

In the notation of (1), we consider the system of algebraic equations

$$(A-I)x = 0 \quad \Big| \quad (A-2I)^2 x = 0$$

in order to determine the subspaces X_1 and X_2 of E_3.

Taking these in succession, we have first

$$(A-I)x = \begin{pmatrix} 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} x = 0 \quad \text{or} \quad \begin{aligned} 2x_1 - x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \\ x_1 - x_2 + x_3 &= 0 \end{aligned}$$

Thus X_1 is the subspace spanned by the vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ with

$x_1 = 0$, $x_2 = x_3$.

and clearly $\dim X_1 = 1$ $(3 - \underbrace{2}_{\text{rank of } (A-I)} = 1)$.

Next

$$(A-2I)^2 x = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix} x = 0 \quad \text{or} \quad \begin{aligned} -x_1 + x_2 &= 0 \\ -x_1 + x_2 &= 0 \end{aligned}$$

Thus X_2 is the subspace spanned by vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ with

$x_1 = x_2$ and x_3 arbitrary.

clearly $\dim X_2 = 2$ ($3 - 1 = 2$, with 1 = rank of $(A-2I)^2$)

We now wish to find vectors $v_1 \in X_1$, $v_2 \in X_2$ such that we can write the initial vector η as

$$\eta = v_1 + v_2$$

since $v_1 \in X_1$, $v_1 = \begin{pmatrix} 0 \\ \alpha \\ \alpha \end{pmatrix}$ for some scalar α, and since

$v_2 \in X_2$, $v_2 = \begin{pmatrix} \beta \\ \beta \\ \gamma \end{pmatrix}$ for some scalars β, γ.

Therefore

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \\ \alpha \end{pmatrix} + \begin{pmatrix} \beta \\ \beta \\ \gamma \end{pmatrix}$$

So that $\beta = \eta_1$, $\alpha + \beta = \eta_2$, $\alpha + \gamma = \eta_3$.

Solving these equations for α, β and γ, we find that

$\alpha = \eta_2 - \eta_1$, $\beta = \eta_1$, $\gamma = \eta_3 - \eta_2 + \eta_1$ and

$$v_1 = \begin{pmatrix} 0 \\ \eta_2 - \eta_1 \\ \eta_2 - \eta_1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} \eta_1 \\ \eta_1 \\ \eta_3 - \eta_2 + \eta_1 \end{pmatrix}$$

Thus by the formula $\circledast$, we find that the solution $\underline{\Phi}(t)$ such that $\underline{\Phi}(0) = \eta$ is given by

$$\underline{\Phi}(t) = e^{t} v_1 + e^{2t}\left[I + (A - 2I)\right] v_2$$

$$\underline{\Phi}(t) = e^{t}\begin{pmatrix} 0 \\ \eta_2 - \eta_1 \\ \eta_2 - \eta_1 \end{pmatrix} + e^{2t}\left(I + \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 0 \end{pmatrix} t \right) \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 - \eta_2 + \eta_1 \end{pmatrix}$$

$$= e^{t}\begin{pmatrix} 0 \\ \eta_2 - \eta_1 \\ \eta_2 - \eta_1 \end{pmatrix} + e^{2t}\begin{pmatrix} 1+t & -t & t \\ 2t & 1-2t & t \\ t & -t & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 - \eta_2 + \eta_1 \end{pmatrix}$$

To find a fundamental matrix, we proceed as in the previous example.

Putting η successively equal to $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in this formula, we obtain the three linearly independent solutions that we use as columns of the matrix

$$\underline{\Phi}(t) = e^{At} = \begin{pmatrix} (1+t)e^{2t} & -te^{2t} & te^{2t} \\ -e^{t} + (1+t)e^{2t} & e^{t} - te^{2t} & te^{2t} \\ -e^{t} + e^{2t} & e^{t} - e^{2t} & e^{2t} \end{pmatrix}$$

Hence $\underline{\Phi}(t)$ is the fundamental matrix that reduces to the identity matrix when $t = 0$.

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Ex :- Find a fundamental matrix for the system $y' = Ay$ with

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

Recall that we have shown previously that

$$(A-4I)^3 = 0$$

So that $(A-4I)^3 x = 0$, for any vector x in E_5 and the initial vector remains arbitrary.

since there is only one eigenvalue ($\lambda = 4$), only the subspace $X_1 = E_5$ is relevant and we have from ⊛

$$\underline{\Phi}(t) = e^{4t}\left[I + (A-4I)t + \frac{t^2}{2!}(A-4I)^2\right]\eta$$

Therefore

$$\underline{\Phi}(t) = e^{4t}\left[I + \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \frac{t^2}{2!}\right]\eta$$

Again letting η successively assume the values

$$\eta_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \eta_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \eta_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

in the above formula, the resulting solutions will be linearly independent and can be used a columns of a fundamental matrix.

Thus

$$\underline{\Phi}(t) = e^{4t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & 0 & 0 \\ 0 & 1 & t & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is a fundamental matrix.

Exercises :-

Find a fundamental matrix for each of the following systems $y' = Ay$ having the coefficient matrix given. Also find a particular solution satisfying the given initial condition.

① $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$; $\eta = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

② $A = \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}$; $\eta = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

③ $A = \begin{pmatrix} 1 & 0 & 3 \\ 8 & 1 & -1 \\ 5 & 1 & -1 \end{pmatrix}$; $\eta = \begin{pmatrix} 0 \\ -2 \\ -7 \end{pmatrix}$

④ $A = \begin{pmatrix} 3 & -1 & -3 & 2 \\ 2 & 3 & -2 & -4 \\ 2 & -1 & -3 & 2 \\ 1 & 2 & -1 & -3 \end{pmatrix}$; $\eta = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

Hint: the characteristic polynomial is $|A - \lambda I| = (\lambda-1)^2(\lambda+1)^2$.

Remark :-

We may find a particular solution of the nonhomogenous system

$$y' = Ay + g(t)$$

where A is a constant matrix and g is a given continuous function.

Recall that the solution of the nonhomogenous system is

$$\phi(t) = \Phi(t)\Phi^{-1}(t_0)c_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)B(s)\,ds$$

since $\Phi(t) = e^{At}$, when A is constant matrix

so, we have $\Phi^{-1}(s) = \exp(-sA)$, $\Phi(t)\Phi^{-1}(s) = \exp[(t-s)A]$

if the initial condition is $\Phi(t_0) = \eta$.

$$\Phi(t)\,\Phi^{-1}(t_0)\,\eta = \exp[(t-t_0)A]\,\eta$$

and the solution of $y' = Ay + g(t)$ is

$$\boxed{\phi(t) = \exp[(t-t_0)A]\eta + \int_{t_0}^{t} \exp[(t-s)A]\, g(s)\, ds, \quad -\infty < t < \infty} \qquad --(5)$$

($\Phi(t) = e^{At}$ is the fundamental matrix of $y' = Ay$).

Ex :- Solve the intial value problem

$$y' = Ay + g(t)$$

where $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$, and where $g(t) = \begin{pmatrix} e^{3t} \\ 1 \end{pmatrix}$, with the initial condition $\phi(0) = \eta$.

Solu :- We have been proved previously that

$$\Phi(t) = e^{At} = e^{3t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}$$

$$\Phi(t)\Phi^{-1}(s) = \exp[(t-s)A] = \exp[3(t-s)]\begin{pmatrix} 1-(t-s) & t-s \\ -(t-s) & 1+(t-s) \end{pmatrix}$$

$$\exp[(t-s)A]\,g(s) = e^{3t}\begin{pmatrix} 1-(t-s)+e^{-3s}(t-s) \\ -(t-s)+e^{-3s}(1+t-s) \end{pmatrix}$$

Therefore

$$\phi(t) = e^{3t}\begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix}\eta + e^{3t}\int_0^t \begin{pmatrix} 1-(t-s)+e^{-3s}(t-s) \\ -(t-s)+e^{-3s}(1+t-s) \end{pmatrix} ds \quad \#$$

Similarity of Matrices and the Jordan Canonical Form:

Def:- A matrix A is said to be similar to B if there exist a nonsingular matrix M such that $B = MAM^{-1}$.

Theorem :- « Jordan »

Any 2×2 matrix A is similar to a matrix J of one of the following forms:-

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

Remark :-

If $A \sim B$, then A and B have the same eigenvalues.

The proof of Jordan theorem:-

recall that

$\det(A-\lambda I)=0$ implies that $\lambda^2 - tr(A)\lambda + \det(A) = 0$

then

$$\lambda = \frac{tr(A) \mp \sqrt{(trA)^2 - 4\det(A)}}{2}, \text{ let } \Delta = (tr(A))^2 - 4\det A$$

if $\Delta > 0 \longrightarrow \lambda_1 \neq \lambda_2$ reals

$\Delta = 0 \longrightarrow \lambda_1 = \lambda_2$ reals

$\Delta < 0 \longrightarrow \lambda = \alpha \mp i\beta$, $\alpha = \frac{tr(A)}{2}$, $\beta = \frac{\sqrt{\Delta}}{2}$

Case1 :- A has two real values $\lambda_1 \neq \lambda_2$, $(\lambda_1 > \lambda_2)$

let $M = (m_1 \quad m_2)$ s.t $Am_1 = \lambda_1 m_1$, $Am_2 = \lambda_2 m_2$

$$AM = A(m_1 \quad m_2) = (Am_1 \quad Am_2)$$
$$= (\lambda_1 m_1 \quad \lambda_2 m_2)$$
$$= (m_1 \quad m_2)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Therefore

$$AM = MJ, \quad J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Hence $A = MJM^{-1}$, therefore A is similar to J.

Case2 A has one real eigenvalue λ

If A is a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then it's of the form given in case1 above.

when A is not diagonal matrix, then dim (eigenspace) = 1

let $M = (m_1 \quad m_2)$

m_1 is an eigenvector

m_2 is any vector, independent of m_1 s.t:

$$AM = A(m_1 \quad m_2) = (Am_1 \quad Am_2)$$
$$= (\lambda m_1 \quad Am_2)$$
$$= M(\lambda e_1 \quad M^{-1}Am_2), \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

since $M^{-1}AM$ and A have the same eigenvalues and

$$M^{-1}AM = \begin{pmatrix} \lambda & c \\ 0 & \lambda \end{pmatrix}, \text{ therefore } M^{-1}Am_2 = \begin{pmatrix} c \\ \lambda \end{pmatrix}$$

Also $M\lambda e_1 = M\begin{pmatrix} \lambda \\ 0 \end{pmatrix} = (m_1 \quad m_2)\begin{pmatrix} \lambda \\ 0 \end{pmatrix} = \lambda m_1$

So, we have

$$AM = M\begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix}$$

Put $M_1 = M\begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$

Therefore

$$M_1^{-1} A M_1 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad A M_1 = M_1 J$$

where $M_1 = (m_1 \; m_2)\begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$, $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

(m_1: eigenvalue)

Case 3 :- A has two complex eigenvalues $\alpha \mp i\beta$

$$M = \begin{pmatrix} a_{11} - \alpha & -\beta \\ a_{21} & 0 \end{pmatrix}, \quad J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

$$\begin{pmatrix} a_{11} - \alpha & -\beta \\ a_{21} & 0 \end{pmatrix}\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} - \alpha & -\beta \\ a_{21} & 0 \end{pmatrix}$$

s.t: $\alpha = \dfrac{a_{11} + a_{22}}{2}$, $\beta = \dfrac{\sqrt{4(a_{11}a_{22} - a_{12}a_{21}) - (a_{11} + a_{22})}}{2}$

Examples :-

① $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$

$\mathrm{tr}A = 1 + 2 = 3$, $\det A = 2$

$\therefore \lambda = \dfrac{3 \mp \sqrt{9 - 8}}{2}$, $\lambda_1 = 2$, $\lambda_2 = 1$

To find m_1 and m_2

$$\begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \quad -x - 3y = 0 \rightarrow x = -3y$$

Therefore $m_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$-3y = 0 \Rightarrow y = 0$

$2y = y \Rightarrow y = 0$

$\therefore m_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Hence $M = (m_1\ m_2) = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}$

check / H.w $AM = MJ$, $A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix}$, $M = \begin{pmatrix} -3 & 1 \\ 1 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

(2) $A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$

$\text{tr}A = 4$, $\det A = 4$, $\therefore \lambda = \dfrac{4 \mp \sqrt{16-16}}{2} = 2$

$$\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$-x - y = 0 \rightarrow x = -y$, Therefore $m_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$x + y = 0 \rightarrow x = -y$

let $m_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

To find c / $M^{-1}AM = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$ → c

$\therefore M_1 = \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{M}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$

$M_1 J = AM_1$

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

Remark

We can Calculate $e^{At} = \Phi(t)$, using Jordan, theorem in the following way:

$$A = MJM^{-1}$$

$$At = MJtM^{-1}$$

Therefore

$$e^{At} = Me^{Jt}M^{-1} \quad / \left(\underbrace{e^{TAT^{-1}} = Te^{A}T^{-1}}_{\text{we prove it previously}} \right)$$

if $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $e^{Jt} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$

$J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, then $e^{Jt} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix}$

$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then $e^{Jt} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$

$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, then $e^{Jt} = e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix}$

$\#$

2. phase plane :-

let $\dot{x} = Ax$, $x \in \mathbb{R}^2$, $A_{2\times 2}$

if $\phi(t) = \begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix}$, $\phi : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}^2$ is a solution

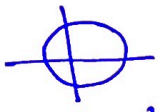
then $\{ (\phi_1(t), \phi_2(t)) : t \in I \} \subseteq \mathbb{R}^2$ is called orbit of ϕ

or trajectory or path.

Example :-

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$

$\phi(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$, $O(\phi) = \{ (\sin t, \cos t) : t \in \mathbb{R} \}$



$x_1^2 + x_2^2 = 1$

$x_1 = \cos t$

$x_2 = \sin t$

$\psi(t) = \begin{pmatrix} \sin(t + \pi/2) \\ \cos(t + \pi/2) \end{pmatrix}$

$O(\psi) = \{ (\sin(t+\pi/2), \cos(t+\pi/2)) \}$

Also it is a circle with radius = 1, but the difference is that the second ones start from the time $t - \pi/2$ while the first ones start from $t = 0$.

Now, we want to sketch the orbit of the equation without solving it.

Suppose $\dot{x} = f(x)$, $x \in \mathbb{R}^2$

f is C^1 map on an open set D.

∴ autonomous equation

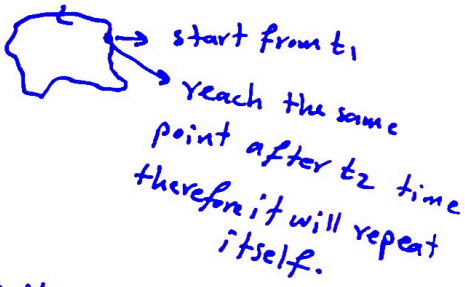
Theorem :- If

① If $\phi(t)$ is a solution then $\phi(t+c)$ is also a solution for any c; the solutions have the same orbit.

② If $\phi(t)$ is a Solu. and $\phi(t_1) = \phi(t_2)$, then

$$\phi(t + t_2 - t_1) = \phi(t)$$

→ the solution will repeat itself within $t_2 - t_1$ time



③ If $x_0 \in D$, then there exist a unique orbit C s.t: $x_0 \in C$.

Proof:- ① let $\psi(t) = \phi(t+c)$. ϕ is defined on I

ψ is defined on $\{t-c : t \in I\} = J$.

$$\dot{\psi} = \dot{\phi}(t+c) = f(\phi(t+c)) = f(\psi(t))$$

$O(\phi) = \{\phi(t) : t \in I\}$, $O(\psi) = \{\phi(t+c) : t \in J\}$

therfore ψ is also a solution

② let $\psi(t) = \phi(t+t_2-t_1)$

$\psi(t)$ is a solution

$\psi(t_1) = \phi(t_2) = \phi(t_1)$

$\therefore \psi = \phi$

Therefore $\phi(t+t_2-t_1) = \phi(t)$

③ If ψ and ϕ are two solutions s.t. $\phi(t_1) = x_0$, $\psi(t_2) = x_0$

let $\psi_1(t) = \psi(t+t_2-t_1)$

ψ_1 is a solution

$\psi_1(t_1) = \psi(t_2) = x_0 = \phi(t_1)$

Therefore $\psi_1 = \phi_1$ implies that $\psi(t+t_2-t_1) = \phi(t)$

Hence ψ and ϕ have the same orbit. //

The phase planes of linear systems:-

$\dot{x} = Ax$, $x \in \mathbb{R}^2$

$\exists\ J, M$ s.t $A = MJM^{-1}$, $M^{-1}AM = J$

let $y = M^{-1}x$,

$\dot{y} = M^{-1}\dot{x} = M^{-1}Ax = M^{-1}AMy$

$= Jy$

So, we have

$\dot{x} = Ax$, $A = MJM^{-1}$

$y = M^{-1}x$, $x = My$

$\boxed{\dot{y} = Jy}$

Now, we start to sketch $\dot{y} = Jy$, using the fourth cases of J:

$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 > \lambda_2$; $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ $\beta > 0$

① $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 > \lambda_2$, $\lambda_1 > 0$, $\lambda_2 > 0$

$$\dot{y} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

therefore

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{aligned} \dot{y}_1 &= \lambda_1 y_1 \\ \dot{y}_2 &= \lambda_2 y_2 \end{aligned}$$

Hence

$$y_1 = c_1 e^{\lambda_1 t}$$
$$y_2 = c_2 e^{\lambda_2 t}$$

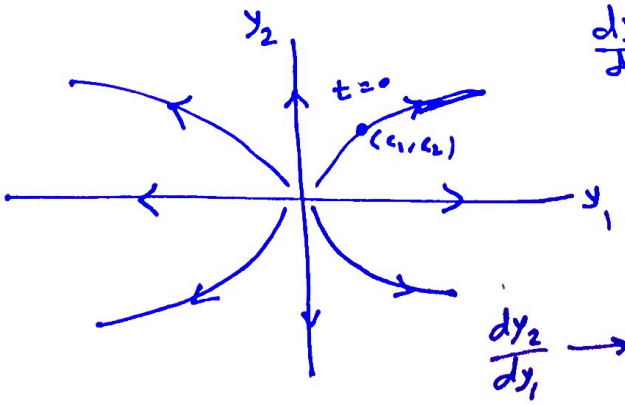
$$\frac{y_1}{c_1} = e^{\lambda_1 t} \rightarrow \left(\frac{y_1}{c_1}\right)^{1/\lambda_1} = e^{t}$$

if $\lambda_1 > \lambda_2 > 0$

$$y_2 = c_2 \left(\frac{y_1}{c_1}\right)^{\frac{\lambda_2}{\lambda_1}} = \frac{c_2}{c_1^{\lambda_2/\lambda_1}} y_1^{\lambda_2/\lambda_1} \qquad 1 > \frac{\lambda_2}{\lambda_1} > 0$$

$$\frac{dy_2}{dy_1} = \frac{dy_2/dt}{dy_1/dt} = \frac{c_2 \lambda_2 e^{\lambda_2 t}}{c_1 \lambda_1 e^{\lambda_1 t}} = \frac{c_2 \lambda_2}{c_1 \lambda_1} e^{(\lambda_2 - \lambda_1)t}$$

($\lambda_2 - \lambda_1$: negative)



$t \to \infty$

$\frac{dy_2}{dy_1} \to 0$

$y_1 \to \infty$

$y_2 \to \infty$

$t \to -\infty$

$\frac{dy_2}{dy_1} \to \infty$

$y_1 \to 0$

$y_2 \to 0$

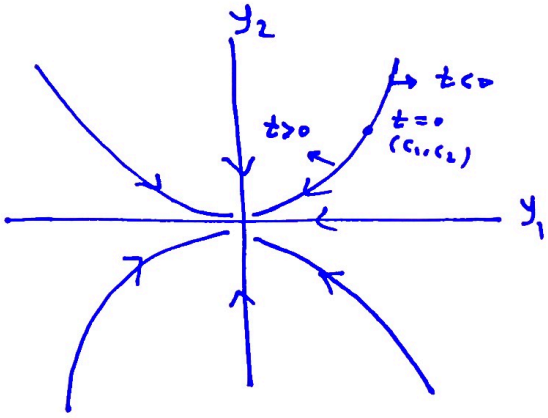
if $\lambda_2 < \lambda_1 < 0$. then $\Rightarrow \frac{\lambda_1}{\lambda_2} > 0$

$$\frac{y_2}{c_2} = e^{\lambda_2 t} \rightarrow \left(\frac{y_2}{c_2}\right)^{\lambda_1} = e^{\lambda_1 \lambda_2 t} = \left(\frac{y_1}{c_1}\right)^{\lambda_2}$$

therefore $\frac{y_2}{c_2} = \left(\frac{y_1}{c_1}\right)^{\frac{\lambda_2}{\lambda_1}}$, $\left(\frac{y_2}{c_2}\right)^{\frac{\lambda_1}{\lambda_2}} = \frac{y_1}{c_1}$

Hence $y_1 = c_1 \left(\frac{y_2}{c_2}\right)^{\lambda_1/\lambda_2}$, like $x = c\sqrt{y}$

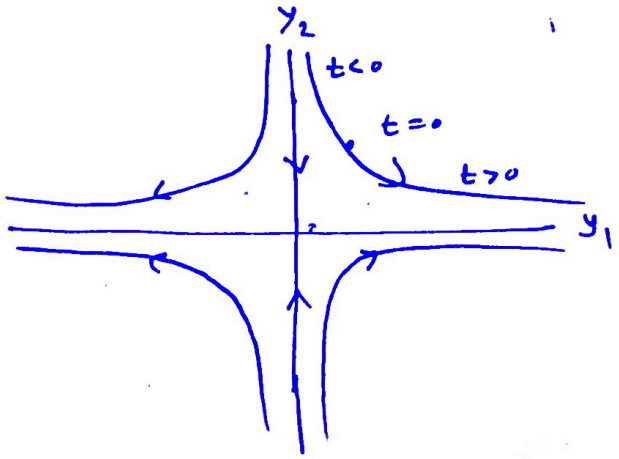
$\lambda_1 < 0$, $\lambda_2 < 0$, $\frac{dy_2}{dy_1} = \frac{\lambda_2 c_2}{\lambda_1 c_1} e^{\underbrace{(\lambda_2 - \lambda_1)}_{\rightarrow \text{negative}} t}$



$t \rightarrow \infty$
$\frac{dy_2}{dy_1} \rightarrow 0$
$y_1 \rightarrow 0$
$y_2 \rightarrow 0$

$t \rightarrow -\infty$
$\frac{dy_2}{dy_1} \rightarrow \infty$
$y_1 \rightarrow \infty$
$y_2 \rightarrow \infty$

if $\lambda_1 > 0$, $\lambda_2 < 0$



(2) $J = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

$\dot{y} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} y \quad , \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow \begin{aligned} \dot{y}_1 &= \lambda y_1 \\ \dot{y}_2 &= \lambda y_2 , \end{aligned}$$

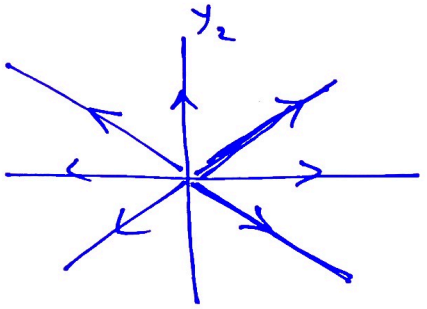
$$\therefore y_1 = c_1 e^{\lambda t}$$

$$y_2 = c_2 e^{\lambda t}$$

if $\lambda > 0$

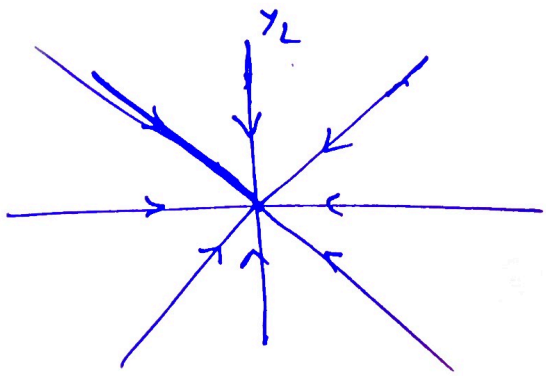
$$\frac{y_2}{y_1} = \frac{c_2}{c_1} = c$$

$$\frac{dy_2}{dy_1} = \frac{\lambda c_1 e^{\lambda t}}{\lambda c_2 e^{\lambda t}} = \frac{c_1}{c_2} = \text{constant}$$



$\lambda > 0$

if $\lambda < 0$



$\lambda < 0$

③ $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\dot{y} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} y$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} y$$

$$\dot{y}_1 = \lambda y_1 + y_2$$

$$\dot{y}_2 = \lambda y_2 \Rightarrow y_2 = c_1 e^{\lambda t}$$

$$\dot{y}_1 = \lambda y_1 + c_1 e^{\lambda t}$$

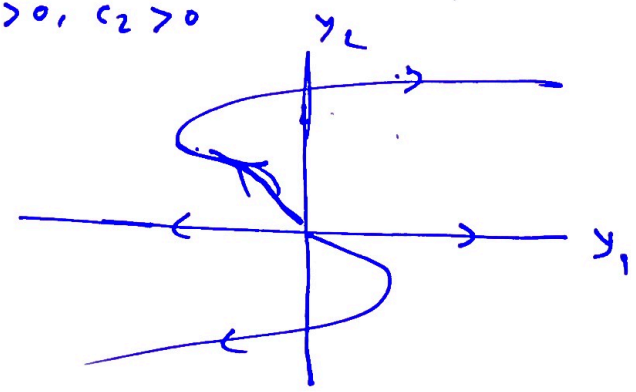
$$\dot{y}_1 - \lambda y_1 = c_1 e^{\lambda t}$$

$$y_1 = e^{-(-\lambda t)} \int e^{-\lambda t} c_1 e^{\lambda t} \, dt$$

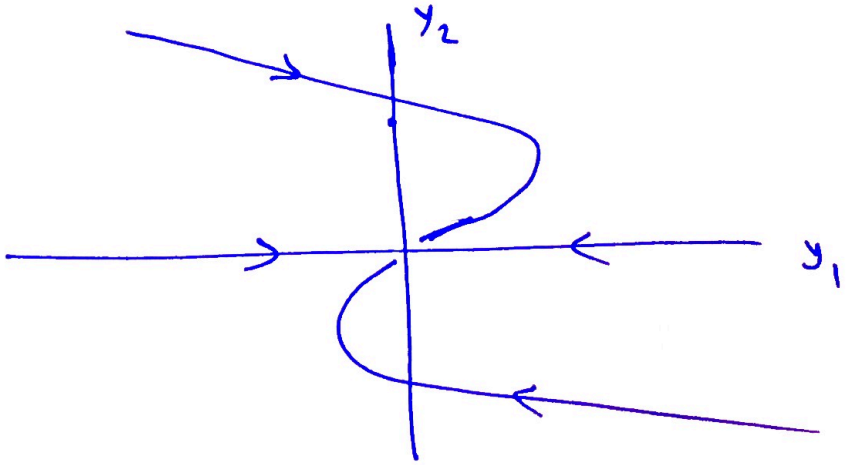
$$y_1 = e^{\lambda t} [c_1 t + c_2]$$

$$y_2 = c_2 e^{\lambda t}$$

if $\lambda > 0$, $c_1 > 0$, $c_2 > 0$



if $\lambda < 0$



(4) $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $\beta > 0$

$$\dot{y} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} y$$

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\dot{y}_1 = \alpha y_1 - \beta y_2$$
$$\dot{y}_2 = \beta y_1 + \alpha y_2$$

let $y_1 = r\cos\theta$ $\qquad r^2 = y_1^2 + y_2^2$

$y_2 = r\sin\theta$ $\qquad \theta = \tan^{-1}\frac{y_2}{y_1}$

$$r\dot{r} = y_1\dot{y}_1 + y_2\dot{y}_2$$
$$= y_1(\alpha y_1 - \beta y_2) + y_2(\beta y_1 + \alpha y_2)$$
$$= \alpha y_1^2 + \alpha y_2^2 = \alpha r^2$$

$$\therefore \dot{r} = \alpha r \rightarrow \boxed{r = c_1 e^{\alpha t}}$$

$$\dot{\theta} = \frac{1}{1 + \left(\frac{y_2}{y_1}\right)^2} \cdot \frac{y_1\dot{y}_2 - y_2\dot{y}_1}{y_1^2}$$

$$= \frac{y_1^2 y_2 (\beta y_1 + \alpha y_2) - y_2(\alpha y_1 - \beta y_2)}{(y_1^2 + y_2^2) y_1^2}$$

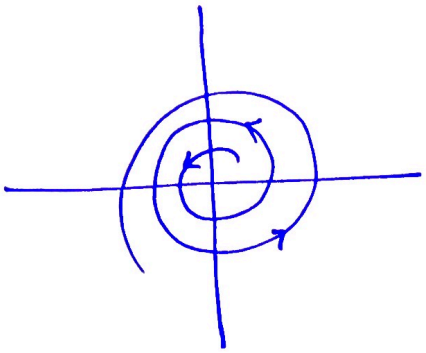
$$\dot{\theta} = \frac{\beta y_1^2 + \beta y_2^2}{r^2} = \frac{\beta r^2}{r^2} = \beta$$

Therefore

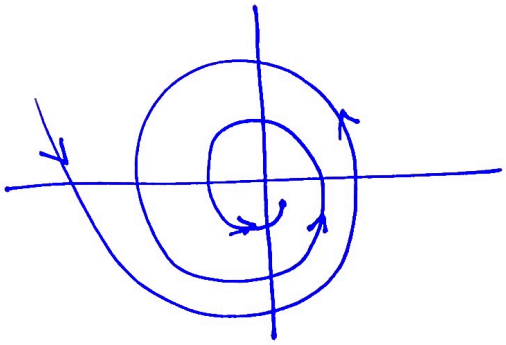
$$\theta = \beta t + c_2$$

$$\boxed{\begin{aligned} r &= c_1 e^{\alpha t} \\ \theta &= \beta t + c_2 \end{aligned}}$$

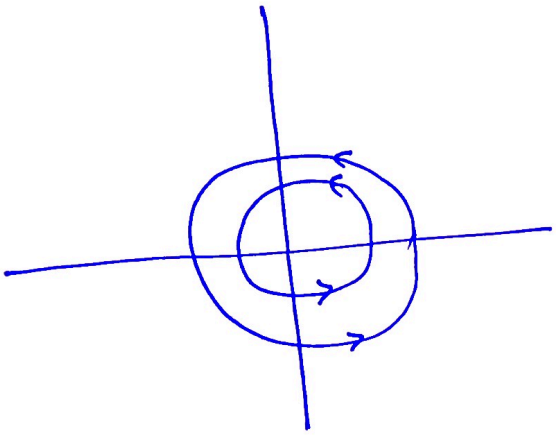
$\alpha > 0$



$\alpha < 0$



$\alpha = 0$



center

Example:-

Sketch the phase plane of

$$\dot{x} = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} x$$

$trA = 3$, $detA = 2$, $\Delta = 9 - 8 = 1 > 0$

$$\lambda = \frac{3 \mp 1}{2} = \frac{3}{2} \mp \frac{1}{2} = \underset{\lambda_1}{2}, \underset{\lambda_2}{1}$$

To find m_1 and m_2

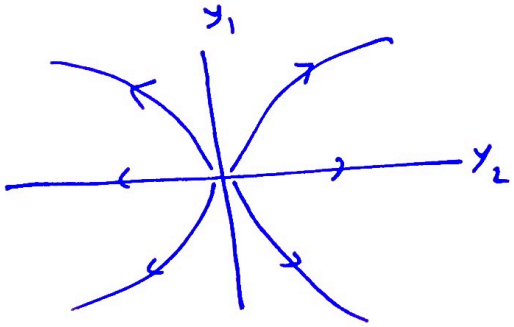
$$\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

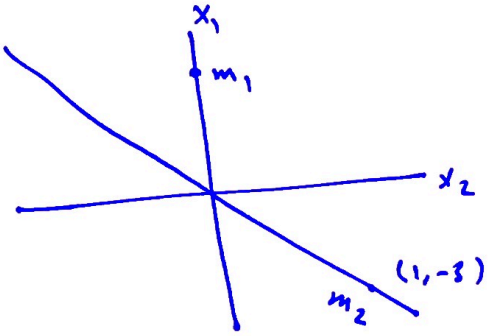
$$-x = 0 \\ 3x = 0 \Rightarrow x = 0 \Rightarrow m_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

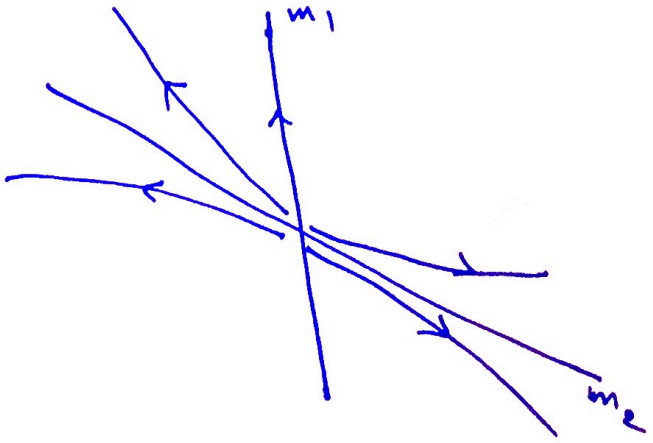
$$\begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$x = x$$

$$3x + 2y = y \Rightarrow 3x + \beta = 0 \Rightarrow m_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$







Example :- sketch the phase plane of

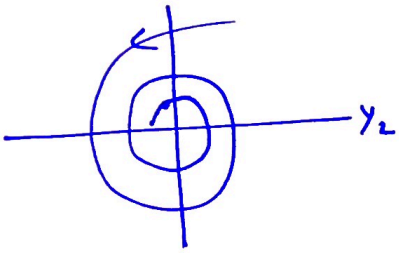
$$\dot{x} = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} x$$

$\text{tr}A = -2$, $\det A = 5$, $\Delta = 4 - 20 = -16$

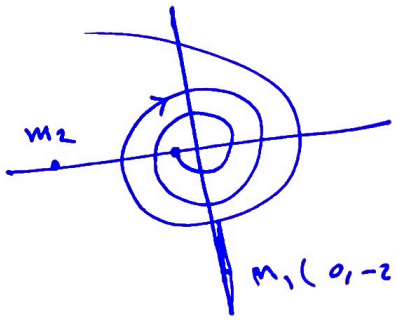
$$\lambda = -1 \mp 2i$$

$$J = \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix} , \quad M = \begin{pmatrix} -1+1 & -2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

m_1 m_2



start from y_1 going to y_2



start from m_1 going to m_2

The Existence and Uniqueness of Solutions

An Existence and Uniqueness Theorem

Consider the equation of order one

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

Let ***T*** denote the rectangular region defined by

$|x - x_0| \leq a$ and $|y - y_0| \leq b$

a region with the point (x_0, y_0) at its center. Let the function f in equation (1) and the function $\partial y/\partial x$ be continuous at each point in T. Then there exists an interval, $|x - x_0| \leq h$, and a function $\phi(x)$ that have the following properties:

a) $y = \phi(x)$ is a solution of equation (1) on the interval $|x - x_0| \leq h$.

b) On the interval $|x - x_0| \leq h$, $\phi(x)$ satisfies the inequality $|\phi(x) - y_0| \leq b$.

c) $\phi(x_0) = y_0$.

d) $\phi(x)$ is unique on the interval $|x - x_0| \leq h$ in the sense that it is the only function that has all of the properties (a), (b), and (c).

The interval $|x - x_0| \leq h$ may or may not need to be smaller than the interval $|x - x_0| \leq a$ over which conditions were imposed on f and $\partial f/\partial y$.

The theorem states that if $f(x, y)$ is sufficiently well behaved near the point (x_0, y_0) the differential equation (1) has a solution that passes through the point (x_0, y_0) and that solution is unique near (x_0, y_0).

A proof of this fundamental theorem is presented in the next three sections. In essence the proof involves showing that a certain sequence of functions has a limit and that the limiting function is the desired solution. The sequence considered will be defined as follows:

$$\begin{aligned} y_0(x) &= y_0 \\ y_1(x) &= y_0 + \int_{x_0}^{x} f\big(t, y_0(t)\big)dt \\ y_2(x) &= y_0 + \int_{x_0}^{x} f\big(t, y_1(t)\big)dt \qquad (2) \\ &\vdots \\ y_n(x) &= y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)dt \end{aligned}$$

So that the proof may appear more reasonable, we first consider some examples of the proof for special differential equations.

Example 1

Show that the sequence of functions defined in equations (2) converges to a solution for the initial value problem

$$\frac{dy}{dx} = y, x_0 = 0, y_0 = 1.$$

We find that

$$y_0(x) = 1$$

$$y_1(x) = 1 + \int_0^x dt = 1 + x$$

$$y_2(x) = 1 + \int_0^x (1 + t)dt = 1 + x + \frac{x^2}{2}$$

$$y_3(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right)dt = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$$

From the pattern that is developing, it is easy to conjecture that

$$y_n(x) = \sum_{i=0}^{n} \frac{x^n}{n!}$$

This is easy to prove by induction. Moreover, the limit of this sequence exists for every real number x because the limit is nothing more than the Maclaurin series expansion for e^x, which converges for every x. That is,

$$\phi(x) = \lim_{n\to\infty} y_n(x) = \sum_{i=0}^{n} \frac{x^n}{n!} = e^x$$

It is a simple matter to verify that e^x is a solution to the initial value problem.

Example 2

Find a solution of the initial value problem

$$\frac{dy}{dx} = x^2, x_0 = 2, y_0 = 1.$$

We find that

$$y_0(x) = 1$$

$$y_1(x) = 1 + \int_2^x t^2 dt = \frac{x^3}{3} - \frac{5}{3},$$

$$y_3(x) = 1 + \int_2^x t^2 dt = \frac{x^3}{3} - \frac{5}{3},$$

$$\vdots$$

$$y_n(x) = 1 + \int_2^x t^2 dt = \frac{x^3}{3} - \frac{5}{3}.$$

Clearly, the limit of this sequence is $\frac{x^3}{3} - \frac{5}{3}$, and this function is a solution of the initial value problem.

Exercises

In each of the following exercises, determine the limit of the sequence defined in (2) above.

1- $y' = x, x_0 = 2, y_0 = 1.$ $\quad Ans: \frac{x^2}{2} - 1$

2- $y' = y, \quad x_0 = 0, y_0 = 2.$ $\quad Ans: 2e^x$

3- $y' = 2y, x_0 = 0, y_0 = 1.$ $\quad Ans: e^{2x}$

4- $y' = x + y, x_0 = 0, y_0 = 1.$ $\quad Ans: 2e^x - x - 1$

A Lipschitz Condition

We have assumed in the hypothesis of the foregoing existence theorem that the function f and its derivative $\partial f/\partial y$ are continuous in the rectangle $\boldsymbol{T}$. Thus if (x, y_1) and (x, y_2) are points in $\boldsymbol{T}$, the mean value theorem applies to f as a function of y. Hence there exists a number y^* between y_1 and y_2 such that

$$f(x, y_1) - f(x, y_2) = \frac{df}{dx}(x, y^*)(y_1 - y_2)$$

The assumption that $\partial f/\partial y$ is continuous in $\boldsymbol{T}$ allows us to assert that $\partial f/\partial y$ is bounded there. That is, there exists a number $K > 0$ such that

$$\left|\frac{\partial f}{\partial y}\right| \leq K,$$

for every point in $\boldsymbol{T}$. Since (x, y^*) is in $\boldsymbol{T}$, it follows that

$$|f(x, y_1) - f(x, y_2)| = \left|\frac{df}{dx}(x, y^*)\right| \cdot |y_1 - y_2|$$

$$|f(x, y_1) - f(x, y_2)| \leq K \cdot |y_1 - y_2| \qquad (3)$$

for every pair of points (x, y_1) and (x, y_2) in $\boldsymbol{T}$.

The inequality (3) is called a *Lipschitz condition* for the function f. We have shown that under the hypotheses of our existence theorem, the Lipschitz condition (3) holds for every pair of points (x, y_1) and (x, y_2) in $\boldsymbol{T}$.

In the proof in next Section we shall actually use the Lipschitz condition rather than the hypothesized continuity of $\partial f/\partial y$. Thus, we could restate the existence theorem in terms of condition (3) instead of assuming that $\partial f/\partial y$ is continuous in $\boldsymbol{T}$.

A Proof of the Existence Theorem

One hypothesis of the *existence theorem* is that f is continuous in the rectangle $\boldsymbol{T}$. It follows that f must be bounded in $\boldsymbol{T}$. Let $M > 0$ be a number such that $|f(x, y)| < M$ for every point in $\boldsymbol{T}$, We now take h to be the smaller of the two

numbers a and b/M, and define the rectangle $\boldsymbol{R}$ to be the set of points (x, y) for which

$$|x - x_0| \leq h \qquad \text{and} \qquad |y - y_0| \leq b$$

Clearly, $\boldsymbol{R}$ is a subset of $\boldsymbol{T}$.

As indicated in (*Existence and Uniqueness Theorem*), we now consider the sequence of functions

$$y_n(x) = y_0 + \int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)dt \tag{4}$$

and prove the following lemma:

Lemma 1 If $|x - x_0| \leq h$, then

$$|y_n - y_0| \leq b$$

For $n = 1,2,3, \ldots$

The proof of this lemma will be accomplished by induction. First of all, if $|x - x_0| \leq h$, we have

$$\begin{aligned} |y_n - y_0| &= \left| \int_{x_0}^{x} f(t, y_0)dt \right| \\ &\leq M \left| \int_{x_0}^{x} dt \right| \\ &\leq M|x - x_0| \\ &\leq Mh \leq b \end{aligned}$$

If we now assume that for $|x - x_0| \leq h$, $|y_n - y_0| \leq b$, it follows that the point $[x, y_k(x)]$ is in $\boldsymbol{R}$ so that $\big|f\big(x, y_k(x)\big)\big| \leq M$. Thus

$$\begin{aligned} |y_{k+1} - y_0| &\leq \left| \int_{x_0}^{x} f\big(t, y_k(t)\big)dt \right| \\ &\leq M \left| \int_{x_0}^{x} dt \right| \\ &\leq Mh \leq b \end{aligned}$$

By induction we can now assert the validity of the lemma.

Lemma 1 may be stated in a slightly different way: If $|x - x_0| \leq h$, then the points $[x, y_k(x)]$, $n = 0,1,2, \ldots$, are in $\boldsymbol{R}$. The Lipschitz condition we given before may now be used to deduce the following lemma.

Lemma 2 If $|x - x_0| \leq h$, then

$$\big|f\big(x, y_n(x)\big) - f\big(x, y_{n-1}(x)\big)\big| \leq K \cdot |y_n(x) - y_{n-1}(x)|,$$

for $n = 0,1,2, \ldots$.

We are now in a position to give an inductive proof of still another lemma.

Lemma 3 If $|x - x_0| \leq h$, then

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1}|x - x_0|^n}{n!} \leq \frac{MK^{n-1}h^n}{n!}$$

for $n = 0,1,2, \ldots$.

For the case $n = 1$, we have from the proof of Lemma 1,

$$|y_1(x) - y_0(x)| \leq M|x - x_0|,$$

Assuming that

$$|y_{n-1}(x) - y_{n-2}(x)| \leq \frac{MK^{n-2}|x - x_0|^{n-1}}{(n-1)!} \tag{5}$$

we must now show that

$$|y_n(x) - y_{n-1}(x)| \leq \frac{MK^{n-1}|x-x_0|^n}{n!}.$$

We will prove this for the case $x_0 \leq x \leq x_0 + h$. From Lemma 2 we have

$$|y_n(x) - y_{n-1}(x)| = \left| \int_{x_0}^{x} \left[f\left(t, y_{n-1}(t)\right) - f\left(t, y_{n-2}(t)\right) \right] dt \right|$$

$$\leq \int_{x_0}^{x} \left| f\left(t, y_{n-1}(t)\right) - f\left(t, y_{n-2}(t)\right) \right| dt$$

$$\leq K \int_{x_0}^{x} |y_{n-1}(t) - y_{n-2}(t)| dt$$

Using the hypothesis (5), we conclude that

$$|y_n(t) - y_{n-1}(t)| \leq \frac{MK^{n-1}}{n!} |x - x_0|^n \tag{6}$$

For the case $x_0 - h < x < x_0$, the same type of argument will yield the same result. The proof of Lemma 3 is thus complete.■

To utilize the results of Lemma 3, we now compare the two infinite series

$$\sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)] \quad and \quad \sum_{n=1}^{\infty} \frac{MK^{n-1}h^n}{n!}$$

The second of these series is an absolutely convergent series. Moreover, by Lemma 3, the second series dominates the first series. Hence, by the Weierstrass M test the series

$$\sum_{n=1}^{\infty} [y_n(x) - y_{n-1}(x)] \tag{7}$$

converges absolutely and uniformly on the interval $|x - x_0| \leq h$. If we consider the kth partial sum of the series (7)

$$\sum_{n=1}^{k} [y_n(x) - y_{n-1}(x)] = [y_1(x) - y_0(x)] + [y_2(x) - y_1(x)] + \cdots + [y_k(x) - y_{k-1}(x)]$$

we see that

$$\sum_{n=1}^{k}[y_n(x) - y_{n-1}(x)] = y_k(x)$$

That is, the statement that the series (7) converges absolutely and uniformly is equivalent to the statement that the sequence $y_k(x)$ converges uniformly on the interval

$$|x - x_0| \leq h$$

If we now define

$$\phi(x) = \lim_{n\to\infty} y_n(x)$$

and recall from the definition of the sequence $y_n(x)$ that each $y_n(x)$ is continuous on $|x - x_0| \leq h$, it follows (since the convergence is uniform) that $\phi(x)$ is also continuous and

$$\phi(x) = \lim_{n\to\infty} y_n(x) = y_0(x) + \lim_{n\to\infty}\int_{x_0}^{x} f\big(t, y_{n-1}(t)\big)dt$$

Because of the continuity of f and the uniform convergence of the sequence $y_n(x)$, We may interchange the order of the two limiting processes to show that $\phi(x)$ is a solution of the integral equation

$$\phi(x) = y_0(x) + \int_{x_0}^{x} f\big(t, \phi(t)\big)dt \qquad (7)$$

It follows immediately upon differentiation of equation (7) that $\phi(x)$ is a solution of the differential equation $dy/dx = f(x, y)$ on the interval $|x - x_0| \leq h$. Furthermore, it is clear from equation (7) that $\phi(x_0) = y_0$.

Finally, since we have shown in Lemma 1 that $|y_n(x) - y_0| \leq b$ for each n and for $|x - x_0| \leq h$, it follows that the same inequality must hold for $\phi(x) = \lim_{n\to\infty} y_n(x)$. That is, if $|x - x_0| \leq h$, then $|\phi(x) - y_0| \leq b$.

Thus we have completed the proof of parts (a), (b), and (c) of the existence theorem.

A Proof of the Uniqueness Theorem

We must now show that the function $\phi(x)$ obtained in previous Section is unique.

Suppose there is another function $Y(x)$, such that $dY/dx = f[x, Y(x)]$, $Y(x_0) = y_0$, and $|Y(x) - y_0| \leq b$ for $|x - x_0| \leq h$. Then we may write

$$Y(x) = y_0(x) + \int_{x_0}^{x} f\big(t, Y(t)\big)dt$$

If we compare $Y(x)$ to the function of the sequence $y_n(x)$ of previous Section, we see that

$$|Y(x) - y_n(x)| \leq \left| \int_{x_0}^{x} [f(t, Y(t)) - f(t, y_{n-1}(x))] dt \right| \tag{8}$$

We shall now show that as $n \to \infty$, the integral on the right side of (8) approaches zero for $|x - x_0| \leq h$. It will then follow that $Y(x) = \lim_{n\to\infty} y_n(x)$, so that finally, $Y(x) \equiv \phi(x)$on the interval $|x - x_0| \leq h$.

For any x on the interval $|x - x_0| \leq h$ it is true that $[x, Y(x)]$ and $[x, y_{n-1}(x)]$ are in the rectangle ***R***; hence the Lipschitz condition will allow us to change (8) into

$$|Y(x) - y_n(x)| \leq K \int_{x_0}^{x} |Y(x) - y_{n-1}(x)| dt \tag{9}$$

We now proceed by an inductive proof and limit our attention to values of x greater than x_0. (A similar argument obtains the same result for $x_0 - h \leq x \leq x_0$).

For $n = 1$, we have

$$\begin{aligned} |Y(x) - y_1(x)| &\leq K \int_{x_0}^{x} |Y(x) - y_0(x)| dt \\ &\leq Kb(x - x_0) \end{aligned}$$

We wish also to show that the assumption

$$|Y(x) - y_{n-1}(x)| \leq \frac{K^{n-1} b (x - x_0)^{n-1}}{(n-1)!}$$

leads to the conclusion that

$$|Y(x) - y_n(x)| \leq \frac{K^n b (x - x_0)^n}{n!} \tag{10}$$

This will complete an inductive argument for the relation (10). We have for $x_0 \leq x \leq x_0 + h$

$$\begin{aligned} |Y(x) - y_n(x)| &\leq \int_{x_0}^{x} |f(t, Y(t)) - f(t, y_{n-1}(t))| dt \\ &\leq K \int_{x_0}^{x} |Y(t) - y_{n-1}(t)| dt \\ &\leq \frac{K^n b}{(n-1)!} \int_{x_0}^{x} (t - x_0)^{n-1} dt \\ &\leq \frac{K^n b}{n!} (t - x_0)^n \end{aligned}$$

thus completing the proof of relation (10).

For $|x - x_0| \leq h$ we have, from the inequality (10),

$$|Y(x) - y_n(x)| \leq \frac{K^n b h^n}{n!} \tag{11}$$

As $n \to \infty$ the expression on the right side of relation (11) approaches zero.

Hence it follows that for $|x - x_0| \leq h$, $y_n(x) \to Y(x)$. Thus $Y(x)$ must be the same function $\phi(x)$ that we obtained in previous section. That is, the solution $\phi(x)$ is unique.

Other Existence Theorems

The existence theorem we have proved in the preceding sections for a first-order equation can be extended to equations of higher order. The simplest such extension is to equations of second order that can be written in the form

$$y'' = f(x, y, y') \tag{12}$$

It is natural to expect the theorem to involve continuity requirements on the function f and its partial derivatives. The theorem may be restated as follows:

Theorem .1 If the function of equation (12) and its partial derivatives with respect to y and y' are continuous functions in a region ***T*** defined by

$$|x - x_0| \leq a, \quad |y - y_0| \leq b, \quad |y' - y_0'| \leq c$$

then there exists an interval $|x - x_0| \leq h$ and a unique function $\phi(x)$ such that $\phi(x)$ is a solution of (1) for all x in the interval $|x - x_0| \leq h$, $\phi(x_0) = y_0$, and $\phi'(x_0) = y_0'$.

A proof of this theorem that is quite similar to the privous proof of $y' = f(x, y)$.

The generalization of the theorem to equations of higher order is direct.

Stability of Nonlinear Systems Ordinary Differential Equations

A nonlinear ODE system with n variables, which we will call x_i, n has functions, f_i (t, X), which determine the derivatives at the state $X = (x_1, x_2, \ldots, x_n)^T$ and time t. That is:

$$\frac{dX}{dx} = \begin{bmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \\ \vdots \\ \frac{\partial x_n}{\partial t} \end{bmatrix} = \begin{bmatrix} f_1\,(t, x_1, x_2, \ldots, x_n) \\ f_2\,(t, x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n\,(t, x_1, x_2, \ldots, x_n) \end{bmatrix}$$

Thus, X indicates the state of the system by identifying the value of each variable in the system. If the systems remain in a state indefinitely, then this state is an equilibrium point X_0, of the system. A nonlinear system may have no, one, or multiple equilibrium points. Since systems often have multiple equilibrium points, we refer to the stability of each equilibrium point, rather than to the stability of the entire system.

Thus nonlinear systems may have stable solutions such as stable nodes or foci, or stable limit cycles.

Equilibrium Points and Stability

Because the equilibrium point X_0 of a system is a point where the system is not changing, the derivatives must all be zero. When we set one of the derivatives to zero, we have an algebraic equation: $f_i\,(t, x_1, x_2, \ldots, x_n) = 0$.

We determine the stability of an equilibrium point by linearizing the system about the equilibrium point. This means we find the multidimensional second derivative of the system evaluated at the equilibrium point, which we call the Jacobian.

$$J = \left[\frac{\partial f_i}{\partial x_i}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_i} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

As long as this system is close to the equilibrium point, the Jacobian can be used to predict the behavior of the system in the same way that the matrix A does in the state space form of a linear system. Thus, the eigenvalues of the Jacobian give the stability of this particular equilibrium point in a nonlinear system. We call the process

of finding equilibrium points and determining their analysis 'nonlinear system stability analysis'. By finding equilibrium points and evaluating the Jacobian, we can apply many of the same tools as we learned for linear systems. If we can identify equilibrium points and calculate Jacobians analytically, we have a powerful tool to analyze the system for all parameter values.

Phase portrait of nonlinear ODE's

we introduced earlier is a nonlinear system of ODEs. We will use the Jacobian matrix-based stability analysis for this system.

Example 1:

Consider the nonlinear system

$$\frac{\partial x_1}{\partial t} = x_1 - x_1 x_2$$
$$\frac{\partial x_2}{\partial t} = x_1 x_2 - x_2$$

First, identify all possible steady states ($f_1\ (t, x_1, x_2) = 0;\ f_2\ (t, x_1, x_2) = 0$). One can quickly check that this system of ODE's has two equilibrium points (0,0) and (1,1).

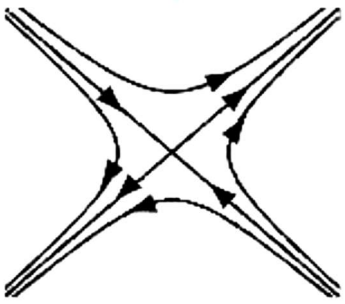
Now we have to construct the Jacobian matrix at each of these two steady states. In general, the Jacobian matrix for this system of ODEs is,

$$J = \begin{bmatrix} \frac{\partial}{\partial x_1}(x_1 - x_1 x_2) & \frac{\partial}{\partial x_2}(x_1 - x_1 x_2) \\ \frac{\partial}{\partial x_1}(x_1 x_2 - x_2) & \frac{\partial}{\partial x_2}(x_1 x_2 - x_2) \end{bmatrix} = \begin{bmatrix} 1 - x_2 & -x_1 \\ x_2 & x_1 - 1 \end{bmatrix}$$

So, J at (0, 0) will be,

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

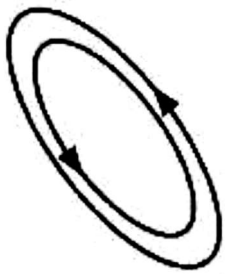
For this matrix, $trJ = 0$ and $detJ = -1$. As $detJ < 0$ this steady state is a saddle ($\lambda_1 = 1, \lambda_2 = -1$).



The Jacobian matrix for the other steady (1, 1) is,

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For this matrix $trJ = 0$ and $detJ = 1$. As $trJ = 0$, $detJ > (trJ)^2/4$, this steady state would be a center type (complex eigenvalue with zero real part $\lambda = \mp i$).



Example 2:

Consider the nonlinear system

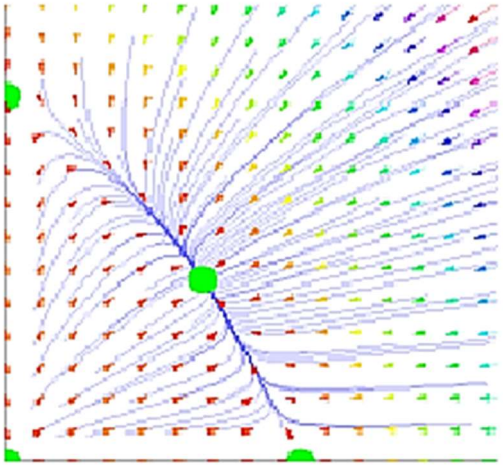
$$\frac{\partial x_1}{\partial t} = x_1(6 - 2x_1 - x_2)$$
$$\frac{\partial x_2}{\partial t} = x_2(4 - x_1 - x_2)$$

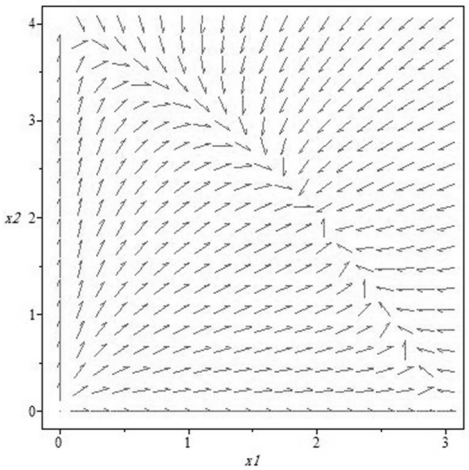
First, identify all possible steady states ($f_1\,(t, x_1, x_2) = 0$; $f_2\,(t, x_1, x_2) = 0$). this system of ODE's has four equilibrium points $X_0 = \{(0,0), (3,0), (0,4), (2,2)\}$.

The Jacobian matrix of the system at the point X_0

$$J = \begin{bmatrix} \frac{\partial}{\partial x_1}(x_1(6 - 2x_1 - x_2)) & \frac{\partial}{\partial x_2}(x_1(6 - 2x_1 - x_2)) \\ \frac{\partial}{\partial x_1}(x_2(4 - x_1 - x_2)) & \frac{\partial}{\partial x_2}(x_2(4 - x_1 - x_2)) \end{bmatrix}$$
$$= \begin{bmatrix} 6 - 2x_1 - x_2 & -x_1 \\ -x_2 & 4 - x_1 - x_2 \end{bmatrix}$$

Equilibrium	Jacobean	Eigenvalues	Nature of equilibrium
(0,0)	$\begin{bmatrix} 6 & 0 \\ 0 & 4 \end{bmatrix}$	$\lambda_1 = 6,\ \lambda_2 = 4$	Unstable
(3,0)	$\begin{bmatrix} -6 & -3 \\ 0 & 1 \end{bmatrix}$	$\lambda_1 = -6,\ \lambda_2 = 1$	saddle
(0,4)	$\begin{bmatrix} 2 & 0 \\ -4 & -4 \end{bmatrix}$	$\lambda_1 = 2,\ \lambda_2 = -4$	saddle
(2,2)	$\begin{bmatrix} -4 & -2 \\ -2 & -2 \end{bmatrix}$	$\lambda_i = -3 \pm \sqrt{5}$	Stable sink





Example 3:

Consider the nonlinear second order differential equation

$$\frac{d^2y}{dt^2} - (1 - y^2)\frac{dy}{dt} + y = 0$$

First of all, transform the nonlinear second order differential equation in to system of first order differential equations

Let $x_1 = y$

$\dot{x}_1 = \dot{y} = x_2$

$\dot{x}_2 = \ddot{y} = (1 - (x_1)^2) \cdot x_2 - x_1$

Then the system of first order differential equations

$\dot{x}_1 = x_2$

$\dot{x}_2 = (1 - (x_1)^2) \cdot x_2 - x_1$

Identify all possible steady states ($f_1\,(t, x_1, x_2) = x_2 = 0$; $f_2\,(t, x_1, x_2) = (1 - (x_1)^2) \cdot x_2 - x_1 = 0$). this system of ODE's has only one equilibrium points $X_0 = (0,0)$

The Jacobian matrix of the system at the point X_0

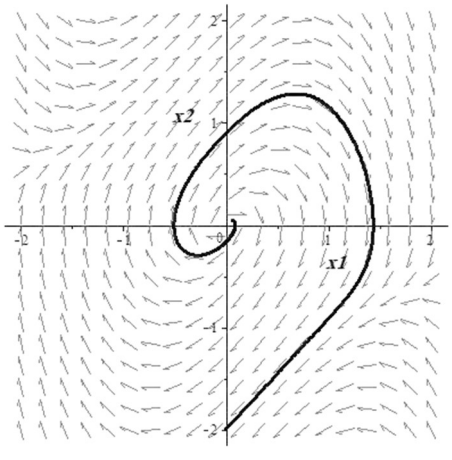
$$J = \begin{bmatrix} \frac{\partial}{\partial x_1} x_2 & \frac{\partial}{\partial x_2} x_2 \\ \frac{\partial}{\partial x_1}\left((1 - (x_1)^2) \cdot x_2 - x_1\right) & \frac{\partial}{\partial x_2}\left((1 - (x_1)^2) \cdot x_2 - x_1\right) \end{bmatrix}$$

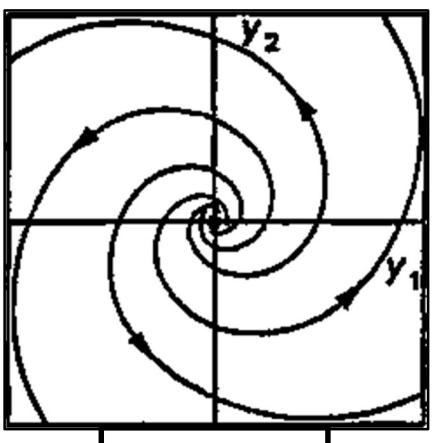
$$= \begin{bmatrix} 0 & 1 \\ -2x_1x_2 - 1 & 1\text{-}(x_1)^2 \end{bmatrix}$$

So, J at (0, 0) will be,

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

For this matrix, $trJ = 1$ and $detJ = 1$. As $detJ > 0$ this steady state is a unstable foci ($\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \lambda_2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$), $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$.





Unstable Foci